

THE TANNO-THEOREM FOR KÄHLERIAN METRICS WITH ARBITRARY SIGNATURE

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ABSTRACT. Considering a non-constant smooth solution f of the Tanno equation on a closed, connected Kähler manifold (M, g, J) with positively definite metric g , Tanno showed that the manifold can be finitely covered by $(\mathbb{CP}(n), \text{const} \cdot g_{FS})$, where g_{FS} denotes the Fubini-Study metric of constant holomorphic sectional curvature equal to 1. The goal of this paper is to give a proof of Tannos Theorem for Kähler metrics with arbitrary signature.

1. INTRODUCTION

1.1. Tanno equation and Main result. Let (M, g, J) be a pseudo-Riemannian Kähler manifold of real dimension $2n$. During the whole paper, we are using tensor notation, for example $T_{j_1 \dots j_q, k}^{i_1 \dots i_p}$ means covariant differentiation of the tensor $T_{j_1 \dots j_q}^{i_1 \dots i_p}$ with respect to the Levi-Civita connection of g . If ω_i is a 1-form on M , then $\bar{\omega}_i = J_i^\alpha \omega_\alpha$. Moreover, the Kählerian 2-form is denoted by the symbol $J_{ij} = g_{i\alpha} J_j^\alpha$. We are interested in the question, if (M, g, J) allows the existence of a non-constant solution f of the equation

$$(1) \quad f_{,ijk} + c(2f_{,k}g_{ij} + f_{,i}g_{jk} + f_{,j}g_{ik} - \bar{f}_{,i}J_{jk} - \bar{f}_{,j}J_{ik}) = 0$$

Originally this equation appeared first in spectral geometry, see [12]. Let $(\mathbb{CP}(n), g_{FS})$ be the complex projective space with Fubini-Study metric g_{FS} of constant holomorphic sectional curvature equal to 1. Then, the eigenfunctions of the Laplacian to the first eigenvalue $-(n+1)$ satisfy (1) with $c = \frac{1}{4}$.

Remark 1. Conversely, contracting (1) with g^{ij} shows that $(\Delta f)_{,k} = -4c(n+1)f_{,k}$, hence

$$\Delta(f + C) = -4c(n+1)(f + C)$$

for any solution f of (1) and some constant C .

The goal of this paper is to give a proof of the following Theorem:

Theorem 1. *Let f be a non-constant smooth function on a closed, connected pseudo-Riemannian Kähler manifold (M^{2n}, g, J) such that the equation (1) is fulfilled for some constant c . Then $c \neq 0$ and $(M^{2n}, 4cg, J)$ can be finitely covered by $(\mathbb{CP}(n), g_{FS}, J_{\text{standard}})$.*

Remark 2. In particular we obtain that on a closed, connected Kähler manifold (M, g, J) such that Cg is not positively definite for any constant $C \neq 0$, any solution of (1) is necessarily a constant. If we assume in addition lightlike completeness, there is a simple and short proof of this statement. Indeed, let f be a solution of equation (1) and let $\gamma : \mathbb{R} \rightarrow M$ be a lightlike geodesic. Restricting (1) to γ yields the ordinary differential equation $f'''(t) = 0$, where $f(t) = f(\gamma(t))$. The general solution $f(t) = At^2 + Bt + C$ must have vanishing constants A and B since γ is complete and f is bounded on the closed manifold M . We obtain that each solution f of equation (1) is constant along any lightlike geodesic. Since two arbitrary points can be connected by a broken lightlike geodesic, we obtain that all solutions of (1) are constant. Theorem 1 generalises this result to the case of a closed, connected lightlike incomplete manifold.

Theorem 1 was proven in [12] for positively definite g and $c > 0$. In this case it is sufficient to require that the manifold is complete. In [4] it was proven that the equation with positively definite g on a closed manifold can not have non-constant solutions for $c \leq 0$.

1.2. The Riemannian analogue: Gallot-Tanno equation. Let us recall that the equation (1) was introduced in [12] as “Kählerization” of

$$(2) \quad f_{,ijk} + c(2f_{,k} \cdot g_{ij} + f_{,i}g_{jk} + f_{,j}g_{ik}) = 0$$

As its Kählerian analogue (1), this equation has its origin in spectral geometry: Consider the Laplacian on the sphere S^n of constant sectional curvature equal to 1. Then the eigenfunctions corresponding to the second eigenvalue $-2(n+1)$ satisfy (2) with $c = 1$, see [12, 2]. Moreover, this equation appeared independently in many other parts of differential geometry. It is known that a non-constant solution of (2) on a Riemannian manifold (M, g) implies the decomposability of the cone $(M' = \mathbb{R}_{>0} \times M, g' = dr^2 + r^2g)$ (decomposability means that (M', g') locally looks like a product manifold), see [2]. In [6, 5] it was shown that this remains to be true in the pseudo-Riemannian situation for a cone over a closed manifold. Furthermore, equation (2) is related to conformal and projective differential geometry (see [3, 12] and [5, 6] for references). The classical result dealing with equation (2) states that on a complete, connected Riemannian manifold, the existence of a non-constant solution of (2) with $c > 0$ implies that the manifold can be covered by the euclidean sphere S^n . This was proven by Gallot and Tanno independently, see [12, 2]. Recently, this result was generalised to the pseudo-Riemannian situation under the additional assumption that the manifold is closed, see [6, 1]. For non-closed manifolds, generalisations of the classical result of Gallot and Tanno for metrics with arbitrary signature were discussed in more detail in [1].

1.3. Organisation of the paper. The goal of this paper is to give a proof of Theorem 1. In Section 2 we consider the case when the constant c in the equation (1) is equal to zero. We show that there are no non-constant solutions of (1) with $c = 0$ on a closed, connected manifold. The relation between non-constant solutions of (1) and holomorph-projective geometry (the Kählerian analogue of projective geometry) will be shown in Section 3. Given a solution of (1), we can construct a solution of a linear PDE system which appears in the theory of holomorph-projectively equivalent Kähler metrics. Going a step further, in Section 4 we assign to each solution of (1) a $(1, 1)$ -tensor on $\widehat{M} := \mathbb{R}^2 \times M$. The family of tensors constructed in this way is invariant with respect to the operation of real polynomials on endomorphisms of the tangent bundle $T\widehat{M}$. This fact allows us to find special solutions of (1) such that the corresponding $(1, 1)$ -tensor acts as a non-trivial projector on $T\widehat{M}$. Using such a solution, the final goal of this Section is to show that the metric g is Riemannian up to multiplication with a constant. The last step in the proof of Theorem 1 is an application of the Theorem proven by Tanno for positive-definite g and will be done in Section 5.

2. THE CASE WHEN $c = 0$

Let us now treat the case when the constant c in the equation (1) is equal to zero. We show

Theorem 2. *Let (M, g, J) be a closed, connected pseudo-Riemannian Kähler manifold and f a solution of $f_{,ijk} = 0$. Then f is a constant.*

Proof. Let p and q be points of M where f takes its maximum and minimum value respectively. Then $f_{,i}(p) = f_{,i}(q) = 0$ and the hessian satisfies $f_{,ij}(p) \leq 0$ and $f_{,ij}(q) \geq 0$. By assumption $f_{,ij}$ is parallel. It follows that $f_{,ij} = 0$ implying that $f_{,i}$ is parallel. Since $f_{,i}(p) = 0$ we obtain that $f_{,i} = 0$ on the whole of M , hence f is a constant. \square

3. SOLUTIONS OF (1) CORRESPOND TO SOLUTIONS OF FROBENIUS SYSTEM

Let f be a non-constant solution of equation (1) on a closed, connected manifold M . By Theorem 2 we obtain $c \neq 0$. The constant c can therefore be included in the metric g without changing the Levi-Civita connection. For simplicity, we denote the new metric again with the symbol g , hence f satisfies equation (1) with $c = 1$. Consider the symmetric $(0, 2)$ -tensor a_{ij} and the function μ defined by

$$(3) \quad a_{ij} := -f_{,ij} - 2fg_{ij} \text{ and } \mu := -2f$$

Then we obtain that the following linear system of PDE's is satisfied:

$$(4) \quad \begin{aligned} a_{ij,k} &= f_i g_{jk} + f_j g_{ik} - \bar{f}_i J_{jk} - \bar{f}_j J_{ik} \\ f_{i,j} &= \mu g_{ij} - a_{ij}, \\ \mu_{,i} &= -2f_i \end{aligned}$$

Indeed, covariantly differentiating $a_{ij} = -f_{,ij} - 2f g_{ij}$ and substituting (1) yields

$$\begin{aligned} a_{ij,k} &= -f_{,ijk} - 2f_{,k} g_{ij} = 2f_{,k} \cdot g_{ij} + f_{,i} g_{jk} + f_{,j} g_{ik} - \bar{f}_{,i} J_{jk} - \bar{f}_{,j} J_{ik} - 2f_{,k} g_{ij} \\ &= f_{,i} g_{jk} + f_{,j} g_{ik} - \bar{f}_{,i} J_{jk} - \bar{f}_{,j} J_{ik}, \end{aligned}$$

which is the first equation in (4). The second and third equations of (4) are equivalent to the definition (3).

Remark 3. The first equation

$$(5) \quad a_{ij,k} = f_i g_{jk} + f_j g_{ik} - \bar{f}_i J_{jk} - \bar{f}_j J_{ik}$$

in (4) is the main equation in holomorph-projective geometry, see for example [11, 9] or [7] for a survey on this topic. Two Kähler metrics g and \bar{g} are said to be *holomorph-projectively equivalent* if their holomorphically-planar curves coincide. Recall that a curve $\gamma : I \rightarrow M$ is called *holomorphically-planar* with respect to the Kähler metric g if there are functions $\alpha, \beta : I \rightarrow \mathbb{R}$ such that $\nabla_{\dot{\gamma}} \dot{\gamma} = \alpha \dot{\gamma} + \beta J \dot{\gamma}$ is satisfied, see [10, 13]. The importance of equation (5) is due to the fact, that the solutions (a_{ij}, f_i) of (5), such that a_{ij} is symmetric and hermitian, are in 1 : 1 correspondence with metrics \bar{g} being holomorph-projectively equivalent to g .

Remark 4. We want to mention that the 1-form f_i corresponding to a solution (a_{ij}, f_i) of (5) always is the differential of a function. Indeed, contracting equation (5) with g^{ij} shows that $f_i = \frac{1}{4} a_{\alpha,i}^\alpha$, in particular $f_{i,j} = f_{j,i}$.

Let us note that the construction (3) is invertible: If there is a solution (a_{ij}, f_i, μ) of (4) we can use the equations in (4) to show that μ always constitutes a solution of equation (1) with $c = 1$. Indeed,

$$\begin{aligned} \mu_{,ijk} &\stackrel{(4)}{=} \nabla_k (-2f_{i,j}) \stackrel{(4)}{=} \nabla_k (-2\mu \tilde{g}_{ij} + 2a_{ij}) \\ &= -2\mu_{,k} \tilde{g}_{ij} + 2a_{ij,k} \stackrel{(4)}{=} -2\mu_{,k} \tilde{g}_{ij} + 2f_i \tilde{g}_{jk} + 2f_j \tilde{g}_{ik} - 2\bar{f}_i \tilde{J}_{jk} - 2\bar{f}_j \tilde{J}_{ik} \\ &\stackrel{(4)}{=} -2\mu_{,k} \tilde{g}_{ij} - \mu_{,i} \tilde{g}_{jk} - \mu_{,j} \tilde{g}_{ik} + \bar{\mu}_{,i} \tilde{J}_{jk} + \bar{\mu}_{,j} \tilde{J}_{ik} \end{aligned}$$

Hence, given a solution (a_{ij}, f_i, μ) of (4) we can define $f := -\frac{1}{2}\mu$ which gives a solution of (1). This definition is the inverse construction to (3) and therefore, the solutions f of (1) and (a_{ij}, f_i, μ) of (4) are in linear 1 : 1 correspondence.

Let us now show the main advantage of working with a system like (4):

Lemma 1. *Let (a_{ij}, f_i, μ) be a solution of the system (4) such that $a_{ij} = 0$, $f_i = 0$, $\mu = 0$ at some point p of the connected Kähler manifold (M, g, J) . Then $a_{ij} \equiv 0$, $f_i \equiv 0$, $\mu \equiv 0$ at all points of M .*

Proof. The system (4) is in Frobenius form, i.e., the derivatives of the unknowns a_{ij}, f_i, μ are expressed as (linear) functions of the unknowns:

$$\begin{pmatrix} a_{ij,k} \\ f_{i,j} \\ \mu_{,i} \end{pmatrix} = F \begin{pmatrix} a_{ij} \\ f_i \\ \mu \end{pmatrix},$$

and all linear systems in the Frobenius form have the property that the vanishing of the solution at one point implies the vanishing at all points. \square

4. IF THERE IS A NON-CONSTANT SOLUTION OF (1) WITH $c = 1$, THEN g IS POSITIVELY DEFINITE

The goal of this section is to prove the following

Theorem 3. *Let (M, g, J) be a closed, connected Kähler manifold. Suppose f is a non-constant solution of (1) with $c = 1$. Then, the metric g is positively definite.*

First we want to show that it is possible to choose one solution of equation (1) such that the corresponding $(1, 1)$ -tensor $a_j^i = g^{i\alpha} a_{\alpha j} = -f_{,j}^i - 2f\delta_j^i$ has a clear and simple structure of eigenspaces and eigenvectors. Evaluating $a_{ij}, f_{,ij}$ and g_{ij} on these eigenspaces will show that g has to be positively definite.

4.1. Matrix of the extended system. In order to find the special solution of (1) mentioned above, we assign to each solution f of (1) a $(1, 1)$ -tensor on the $(2n + 2)$ -dimensional manifold $\widehat{M} = \mathbb{R}^2 \times M$ with coordinates $(\underbrace{x_+, x_-}_{\mathbb{R}^2}, \underbrace{x_1, \dots, x_{2n}}_M)$. For every solution f of equation (1), let us consider the $(2n + 2) \times (2n + 2)$ -matrix

$$(6) \quad L(f) = \left(\begin{array}{cc|ccc} \mu & 0 & f_1 & \dots & f_{2n} \\ 0 & \mu & \bar{f}_1 & \dots & \bar{f}_{2n} \\ \hline f^1 & \bar{f}^1 & & & \\ \vdots & \vdots & & & \\ f^{2n} & \bar{f}^{2n} & & & a_j^i \end{array} \right)$$

where a_{ij} and μ are defined by the equations (3). The matrix $L(f)$ is a well-defined $(1, 1)$ -tensor field on \widehat{M} (in the sense that after a change of coordinates on M the components of the matrix L transform according to tensor rules).

Remark 5. Using (3) we see that the identity transformation of $T\widehat{M}$ corresponds to the function $f = -\frac{1}{2}$, i.e.

$$L\left(-\frac{1}{2}\right) = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & & & \delta_j^i \end{array} \right) = \mathbf{1}$$

Remark 6. The matrix L contains the information on the function f and its first and second derivatives. By equation (3) and Lemma 1, if this matrix is vanishing at some point of \widehat{M} then $f \equiv 0$ on the whole of M . In the next section, we will see that the matrix formalism does have advantages: we will show that the polynomials of the matrix L also correspond to certain solutions of equation (1).

Remark 7. The constructions which were done here are visually similar to those which can be done for equation (2). However, in the non-Kählerian situation the corresponding $(1, 1)$ -tensor on the cone manifold is covariantly constant with respect to the connection induced by the cone metric, see [2, 6, 5]. This is not the case for the extended operator (6) which poses additional difficulties.

4.2. Algebraic properties of L . Obviously, the mapping $f \mapsto L(f)$ applied to arbitrary smooth functions f on M is a linear injective mapping between the space of smooth functions on M and the space of $(1, 1)$ -tensors on \widehat{M} . For two smooth functions F and H on M let us define a new product

$$F * H := -2FH - \frac{1}{2}F_{,\alpha}H^\alpha, \text{ and the } k\text{-fold potency } F^{*k} := \underbrace{F * \dots * F}_{k \text{ times}}$$

The product $*$ is bilinear and commuting but not associative. However, it turns out that the mapping L now preserves the potencies f^{*k} of solutions of (1):

Lemma 2. (1) *Let f be a solution of equation (1) with $c = 1$. Then, for every $k \geq 0$ there exists a solution \tilde{f} such that*

$$L^k(f) = L(\tilde{f}), \text{ where } L^k = \underbrace{L \cdot \dots \cdot L}_{k \text{ times}}.$$

(2) *If f is a solution of (1) with $c = 1$, then $L^k(f) = L(f^{*k})$. In particular, f^{*k} is a solution of (1).*

Proof. (1) Given two solutions f and F of equation (1), we denote by (a_{ij}, f_i, μ) and $(A_{ij}, F_i, \mathcal{M})$ the corresponding solutions of (4), constructed by (3). After direct calculation we obtain for the product of the corresponding matrices $L(f)$ and $L(F)$:

$$(7) \quad L(f) \cdot L(F) = \left(\begin{array}{cc|ccc} \mu\mathcal{M} + f_k F^k & f_k \bar{F}^k & \mu F_1 + f_k A_1^k & \dots & \mu F_{2n} + f_k A_{2n}^k \\ \bar{f}_k F^k & \mu\mathcal{M} + f_k F^k & \mu \bar{F}_1 + \bar{f}_k A_1^k & \dots & \mu \bar{F}_{2n} + \bar{f}_k A_{2n}^k \\ \hline \mathcal{M} f^1 + a_k^1 F^k & \mathcal{M} \bar{f}^1 + a_k^1 \bar{F}^k & & & \\ \vdots & \vdots & & & \\ \mathcal{M} f^{2n} + a_k^{2n} F^k & \mathcal{M} \bar{f}^{2n} + a_k^{2n} \bar{F}^k & & & \end{array} \right) \begin{array}{c} \\ \\ a_k^i A_j^k + f^i F_j + \bar{f}^i \bar{F}_j \\ \\ \end{array}$$

Suppose that

$$(8) \quad \mu F_j + f_k A_j^k = \mathcal{M} f_j + a_j^k F_k \quad \text{and} \quad f^k \bar{F}_k = 0$$

then $L(f) \cdot L(F)$ takes the form (6)

$$(9) \quad L(f) \cdot L(F) = \left(\begin{array}{cc|ccc} \tilde{\mu} & 0 & \tilde{f}_1 & \dots & \tilde{f}_{2n} \\ 0 & \tilde{\mu} & \tilde{f}_1 & \dots & \tilde{f}_{2n} \\ \hline \tilde{f}^1 & \tilde{f}^1 & & & \\ \vdots & \vdots & & & \\ \tilde{f}^{2n} & \tilde{f}^{2n} & & & \end{array} \right) \begin{array}{c} \\ \\ \tilde{a}_j^i \\ \\ \end{array}$$

where $\tilde{a}_j^i = a_k^i A_j^k + f^i F_j + \bar{f}^i \bar{F}_j$, $\tilde{f}_i = \mu F_i + A_i^k f_k$ and $\tilde{\mu} = \mu\mathcal{M} + f_k F^k$. Now we show that \tilde{a}_{ij} , \tilde{f}_i and $\tilde{\mu}$ satisfy (4). In addition, we show that $L(f) \cdot L(F)$ is self-adjoint. Let us check the first equation of (4):

$$\begin{aligned} \tilde{a}_{ij,k} &= (a_{is} A_j^s + f_i F_j + \bar{f}_i \bar{F}_j)_{,k} = a_{is,k} A_j^s + a_i^s A_{sj,k} + f_{i,k} F_j + f_i F_{j,k} + \bar{f}_{i,k} \bar{F}_j + \bar{f}_i \bar{F}_{j,k} \\ &\stackrel{(4)}{=} A_j^s f_i g_{sk} + A_j^s f_s g_{ik} + A_j^s \bar{f}_i J^{s'}_{s'k} + A_j^s \bar{f}_s J^{i'}_{i'k} + a_i^s F_j g_{sk} + a_i^s F_s g_{jk} + a_i^s \bar{F}_j J^{s'}_{s'k} + a_i^s \bar{F}_s J^{j'}_{j'k} \\ &\quad + \mu g_{ik} F_j - a_{ik} F_j + \mathcal{M} g_{jk} f_i - A_{jk} f_i + \mu J^{i'}_{i'k} g_{i'k} \bar{F}_j - J^{i'}_{i'k} a_{i'k} \bar{F}_j + \mathcal{M} J^{j'}_{j'k} g_{j'k} \bar{f}_i - J^{j'}_{j'k} a_{j'k} \bar{f}_i \\ &= g_{ik} (f_s A_j^s + \mu F_j) + g_{jk} (F_s a_i^s + \mathcal{M} f_i) + J^{i'}_{i'k} g_{i'k} (\bar{f}_s A_j^s + \mu \bar{F}_j) + J^{j'}_{j'k} g_{j'k} (\bar{F}_s a_i^s + \mathcal{M} \bar{f}_i) \\ &\stackrel{(8)}{=} \tilde{f}_i g_{jk} + \tilde{f}_j g_{ik} - J_i^\alpha \tilde{f}_\alpha J_{jk} - J_j^\alpha \tilde{f}_\alpha J_{ik} \end{aligned}$$

For the second equation one can calculate:

$$\begin{aligned} \tilde{f}_{i,k} &= (\mu F_i + f_j A_i^j)_{,k} = \mu_{,k} F_i + \mu F_{i,k} + f_{j,k} A_i^j + f^j A_{ij,k} \\ &\stackrel{(4)}{=} -2f_k F_i + \mu \mathcal{M} g_{ik} - \mu A_{ik} + \mu A_{ik} - a_{jk} A_i^j + f^j F_i g_{jk} + f^j F_j g_{ik} + f^j J^{j'}_{j'} J^{i'}_{i'} F_{i'} g_{j'k} + f^j J^{j'}_{j'} J^{i'}_{i'} F_{j'} g_{i'k} \\ &= (\mu \mathcal{M} + f^j F_j) g_{ik} - (f_k F_i + \bar{f}_k \bar{F}_i + A_{ij} a_k^j) + f^j \bar{F}_j J^{i'}_{i'} g_{i'k} \stackrel{(8)}{=} \tilde{\mu} g_{ki} - \tilde{a}_{ki} \end{aligned}$$

As we already stated in Remark 4, since $(\tilde{a}_{ij}, \tilde{f}_i)$ satisfies (5), the 1-form \tilde{f}_i is the differential of a function. It follows that $\tilde{f}_{i,j}$ is symmetric and from the last calculation we obtain that \tilde{a}_{ij} is

symmetric since it is the linear combination of two symmetric tensors. Let us now check the third equation of (4):

$$\begin{aligned}\tilde{\mu}_{,i} &= (\mu\mathcal{M} + f_k F^k)_{,i} = \mu_{,i}\mathcal{M} + \mu\mathcal{M}_{,i} + f_{k,i}F^k + f^k F_{k,i} \\ &\stackrel{(4)}{=} -2f_i\mathcal{M} - 2F_i\mu + F^k(\mu g_{ik} - a_{ik}) + f^k(\mathcal{M}g_{ik} - A_{ik}) = -(\mu F_i + f_k A_i^k) - (\mathcal{M}f_i + F_k a_i^k) \stackrel{(8)}{=} -2\tilde{f}_i\end{aligned}$$

Thus, $(\tilde{a}_{ij}, \tilde{f}_i, \tilde{\mu})$ is a solution of (4).

Now we show that the operator $L(F) = L^k(f)$ satisfies the conditions (8). Since $L^k \cdot L = L \cdot L^k$, using (7) we obtain

$$\mu F_j + f_k A_j^k = \mathcal{M}f_j + a_j^k F_k$$

The last condition will be checked by induction. Suppose $f^i \tilde{F}_i = 0$ then

$$f^i J^{i'}_{,i} \tilde{F}_{i'} = f^i \cdot J^{i'}_{,i} (\mu F_{i'} + f_k A_{i'}^k) = \mu \cdot 0 + f^k (J^{i'}_{,i} A_{ki'}) f^i = 0.$$

From the constructions in (3), it is obvious that $L(\tilde{f}) = L^{k+1}(f)$ where $\tilde{f} = -\frac{1}{2}\tilde{\mu}$. This completes the proof of the first part of Lemma 2.

Part (2) follows immediately from the already proven part: Proceeding by induction, we assume $L(f^{*k}) = L^k(f)$. We already know from the first part of Lemma 2, that $L^{k+1}(f) = L(f) \cdot L^k(f) = L(\tilde{f})$. The solution \tilde{f} of (1) is given by the formula $\tilde{f} = -\frac{1}{2}\tilde{\mu} = -2fF - \frac{1}{2}f_k F^k$, see the equations following formula (9). By assumption $F = f^{*k}$ and therefore $\tilde{f} = f * f^{*k} = f^{*(k+1)}$. \square

Let us consider the natural operation of polynomials with real coefficients on elements of $T_1^1 \widehat{M}$. From Lemma 2, we immediately obtain that the family of (1,1)-tensors $L(f)$, constructed from solutions f of (1) is invariant with respect to this operation:

Corollary 1. *Let f be a solution of (1) with $c = 1$ and $P(t) = c_k t^k + \dots + c_1 t + c_0$ an arbitrary polynomial with real coefficients. Then $P*(f) := c_k f^{*k} + c_{k-1} f^{*(k-1)} + \dots + c_1 f - \frac{1}{2}c_0$ is a solution of (1) with $c = 1$ and $L(P*(f)) = P(L(f))$.*

4.3. There exists a solution f of (1) such that $L(f)$ is a projector. We assume that (M, g, J) is a closed, connected Kähler manifold. Our goal is to show that the existence of a non-constant solution f of (1) with $c = 1$ implies the existence of a solution \tilde{f} of (1) such that the matrix $L(\tilde{f})$ is a non-trivial (i.e. $\neq 0$ and $\neq 1$) projector. (Recall that a matrix L is a *projector*, if $L^2 = L$.) We need

Lemma 3. *Let (M, g, J) be a connected Kähler manifold and f a solution of (1) with $c = 1$.*

Let $P(t)$ be the minimal polynomial of $L(f)$ at the point $\hat{p} \in \widehat{M}$. Then, $P(t)$ is the minimal polynomial of $L(f)$ at every point $\hat{q} \in \widehat{M}$.

Convention. *We will always assume that the leading coefficient of a minimal polynomial is 1.*

Proof. As we have already proved, there exists a solution \tilde{f} such that

$$P(L(f)) = L(\tilde{f}).$$

Since $P(L(f))$ vanishes at the point $\hat{p} = (x_+, x_-, p)$, we obtain that $\tilde{a}_{ij} = 0$, $\tilde{f}_i = 0$ and $\tilde{\mu} = 0$ at p , where $(\tilde{a}_{ij}, \tilde{f}_i, \tilde{\mu})$ denotes the solution of (4) corresponding to \tilde{f} . Then, by Lemma 1, the solution $(\tilde{a}_{ij}, \tilde{f}_i, \tilde{\mu})$ of (4) is identically zero on M . Thus, $P(L(f))$ vanishes at all points of \widehat{M} . It follows, that the polynomial $P(t)$ is divisible by the minimal polynomial of $L(f)$ at \hat{q} . By the same reasoning (interchanging p and q), we obtain that $Q(t)$ is divisible by $P(t)$. Consequently, $P(t) = Q(t)$. \square

Corollary 2. *The eigenvalues of $L(f)$ are constant functions on \widehat{M} .*

Proof. By Lemma 3, the minimal polynomial does not depend on the points of \widehat{M} . Then, the roots of the minimal polynomial are also constant (i.e., do not depend on the points of \widehat{M}). \square

In order to find the desired special solution of equation (1), we will use that M is closed.

Lemma 4. *Suppose (M, g, J) is a closed, connected Kähler manifold. Let f be a non-constant solution of (1) with $c = 1$. Then at every point of \widehat{M} the matrix $L(f)$ has at least two different real eigenvalues.*

Proof. Let (a_{ij}, f_i, μ) denote the solution of (4) corresponding to f . Since M is closed, the function μ admits its maximal and minimal values μ_{\max} and μ_{\min} . Let $p \in M$ be a point where $\mu = \mu_{\max}$. At this point, $\mu_{,i} = 0$ implying $f_i = \bar{f}_i = 0$. Then, the matrix of $L(f)$ at p has the form

$$(10) \quad L(f) = \left(\begin{array}{cc|ccc} \mu_{\max} & 0 & 0 & \dots & 0 \\ 0 & \mu_{\max} & 0 & \dots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & & & a_j^i \end{array} \right)$$

Thus, μ_{\max} is an eigenvalue of $L(f)$ at p and, since the eigenvalues are constant, μ_{\max} is an eigenvalue of $L(f)$ everywhere on M . The same holds for μ_{\min} . Since $f_i \not\equiv 0$, μ is not constant implying $\mu_{\max} \neq \mu_{\min}$. Finally, $L(f)$ has two different real eigenvalues μ_{\max}, μ_{\min} at every point. \square

Remark 8. For further use let us note that in the proof of Lemma 4 we have proved that if $\mu_{,i} = 0$ at a point p then $\mu(p)$ is an eigenvalue of L .

Finally, let us show that there is always a solution of (1) of the desired special kind:

Lemma 5. *Suppose (M, g, J) is a closed and connected Kähler manifold. For every non-constant solution f of (1) with $c = 1$, there exists a polynomial $P(t)$ such that $P(L(f))$ is a non-trivial (i.e. it is neither $\mathbf{0}$ nor $\mathbf{1}$) projector.*

Proof. We take a point $\hat{p} \in \widehat{M}$. By Lemma 4, $L(f)$ has at least two real eigenvalues at the point \hat{p} . Then, by linear algebra, there exists a polynomial P such that $P(L(f))$ is a nontrivial projector at the point \hat{p} . Evidently, a matrix is a nontrivial projector, if and only if its minimal polynomial is $t(t-1)$ (multiplied by any nonzero constant). Since by Lemma 3 the minimal polynomial of $P(L(f))$ is the same at all points, the matrix $P(L(f))$ is a projector at every point of \widehat{M} . \square

Thus, given a non-constant solution of (1) with $c = 1$ on a closed and connected Kähler manifold M , without loss of generality we can think that a solution f of (1) with $c = 1$ is chosen such that the corresponding $(1, 1)$ -tensor $L(f)$ is a non-trivial projector.

4.4. Structure of eigenspaces of a_j^i , if $L(f)$ is a nontrivial projector. Assuming that $L(f)$ is a nontrivial projector, it has precisely two eigenvalues 1 and 0 and the $(2n+2)$ -dimensional tangent space of \widehat{M} at every point $\hat{x} = (x_+, x_-, p)$ can be decomposed into the sum of the corresponding eigenspaces

$$T_{\hat{x}}\widehat{M} = E_{L(f)}(1) \oplus E_{L(f)}(0).$$

The dimensions of $E_{L(f)}(1)$ and $E_{L(f)}(0)$ are even; we assume that the dimension of $E_{L(f)}(1)$ is $2k+2$ and the dimension of $E_{L(f)}(0)$ is $2n-2k$. If (a_{ij}, f_i, μ) is the solution of (4) corresponding to f , μ_{\max} and μ_{\min} are eigenvalues of $L(f)$, see Lemma 4. Since $L(f)$ is a projector we obtain $\mu_{\min} = 0 \leq \mu(x) \leq 1 = \mu_{\max}$ on M . In view of Remark 8, the only critical values of μ are 1 and 0.

Lemma 6. *Let f be a solution of (1) with $c = 1$ such that $L(f)$ is a non-trivial projector. Let (a_{ij}, f_i, μ) be the solution of (4) corresponding to f . Then, the following statements hold:*

- (1) *At the point p such that $0 < \mu < 1$, a_j^i has the following structure of eigenvalues and eigenspaces*
 - (a) *eigenvalue 1 with geometric multiplicity $2k$;*
 - (b) *eigenvalue 0 with geometric multiplicity $(2n-2k-2)$;*
 - (c) *eigenvalue $(1-\mu)$ with multiplicity 2.*
- (2) *At the point p such that $\mu = 1$, a_j^i has the following structure of eigenvalues and eigenspaces:*
 - (a) *eigenvalue 1 with geometric multiplicity $2k$;*

- (b) eigenvalue 0 with geometric multiplicity $(2n - 2k)$;
- (3) At the point p such that $\mu = 0$, a_j^i has the following structure of eigenvalues and eigenspaces:
 - (a) eigenvalue 1 with geometric multiplicity $2k + 2$;
 - (b) eigenvalue 0 with geometric multiplicity $(2n - 2k - 2)$.

Convention. We identify M with the set $(0, 0) \times M \subset \widehat{M}$. This identification allows us to consider $T_x M$ as a linear subspace of $T_{(0,0) \times x} \widehat{M}$: the vector $(v_1, \dots, v_n) \in T_x M$ is identified with $(0, 0, v_1, \dots, v_n) \in T_{(0,0) \times x} \widehat{M}$.

Proof. Consider the point p such that $0 < \mu < 1$. For any vector $v \in E_{L(f)}(1) \cap T_p M$ we calculate

$$(11) \quad L(f)v = \left(\begin{array}{cc|ccc} \mu & 0 & f_1 & \dots & f_{2n} \\ 0 & \mu & \bar{f}_1 & \dots & \bar{f}_{2n} \\ \hline f^1 & \bar{f}^1 & & & \\ \vdots & \vdots & & & \\ f^{2n} & \bar{f}^{2n} & & a_j^i & \end{array} \right) \begin{pmatrix} 0 \\ 0 \\ v^1 \\ \vdots \\ v^{2n} \end{pmatrix} = \begin{pmatrix} f_j v^j \\ \bar{f}_j v^j \\ a_j^i v^j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v^1 \\ \vdots \\ v^{2n} \end{pmatrix}$$

Thus, $v = (v^1, \dots, v^{2n})$ is an eigenvector of a_j^i with eigenvalue 1, hence $E_{L(f)}(1) \cap T_p M$ is contained in the eigenspace E_1 of a_j^i with eigenvalue 1. Similarly, any $v \in E_{L(f)}(0) \cap T_p M$ is an eigenvector of a_j^i with eigenvalue 0 and $E_{L(f)}(0) \cap T_p M$ is contained in the eigenspace E_0 of a_j^i with eigenvalue 0. Note that the dimension of $E_{L(f)}(1) \cap T_p M$ is at least $2k$, and the dimension of $E_{L(f)}(0) \cap T_p M$ is at least $2n - 2k - 2$. Let us now show that f^i and \bar{f}^i are eigenvectors of a_j^i with eigenvalue $(1 - \mu)$. We multiply the first basis vector $(1, 0, \dots, 0)$ by the matrix $L(f)^2 - L(f)$ (which is identically zero) and obtain

$$(12) \quad 0 = (L(f)^2 - L(f)) \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \mu^2 + f_j f^j - \mu \\ f_i f^i \\ \mu f^i + a_j^i f^j - f^i \end{pmatrix}$$

This gives us the necessary equation $a_j^i f^j = (1 - \mu) f^i$. Considering the same procedure for the second basis vector $(0, 1, \dots, 0)$ we obtain that \bar{f}^i is an eigenvector of a_j^i to the eigenvalue $(1 - \mu)$ and hence, the dimension of the eigenspace $E_{1-\mu}$ of a_j^i corresponding to the eigenvalue $1 - \mu$ is at least 2. On the one hand, $\dim E_1 + \dim E_0 + \dim E_{1-\mu}$ is at most $2n$ but on the other hand, $2k \leq \dim(E_{L(f)}(1) \cap T_p M) \leq \dim E_1$, $2n - 2k - 2 \leq \dim(E_{L(f)}(0) \cap T_p M) \leq \dim E_0$ and $2 \leq \dim E_{1-\mu}$ implying that

$$(13) \quad T_p M = E_1 \oplus E_0 \oplus E_{1-\mu}.$$

Furthermore $E_1 = E_{L(f)}(1) \cap T_p M$, $E_0 = E_{L(f)}(0) \cap T_p M$ and $E_{1-\mu} = \text{span}\{f^i, \bar{f}^i\}$ are of dimensions $2k$, $2n - 2k - 2$ and 2 respectively. The proof at the points such that $\mu = 0$ or $\mu = 1$ is similar. \square

4.5. If there exists a solution f of equation (1) such that $L(f)$ is a non-trivial projector, the metric g is positively definite on a closed M . Above we have proven that, given a non-constant solution of (1) with $c = 1$, there always exists a solution f of (1) such that the corresponding matrix $L(f)$ is a non-trivial projector. If (a_{ij}, f_i, μ) is the solution of (4) corresponding to f , this implies that the eigenvalues and the dimension of the eigenspaces of a_j^i are given by Lemma 6. Now we are ready to prove that g is positively definite (as we claimed in Theorem 3).

Let us consider such a solution f and the corresponding solution (a_{ij}, f_i, μ) of (4). We rewrite the second equation in (4) in the form

$$(14) \quad \mu_{,ij} = 2a_{ij} - 2\mu g_{ij}$$

Let p be a point where μ takes its maximum value 1. As we have already shown $f^i(p) = 0$ and the tangent space $T_p M$ is equal to the direct sum of the eigenspaces:

$$T_p M = E_1 \oplus E_0$$

Consider the restriction of (14) to E_0 . Since the restriction of the bilinear form a_{ij} to E_0 is identically zero, the equation takes the form:

$$\mu_{,ij}|_{E_0} = -2 g_{ij}|_{E_0}.$$

But $\mu_{,ij}$ is the Hessian of μ at the maximum point p . Then, it is non-positively definite. Hence, the non-degenerate metric tensor g_{ij} is positively definite on E_0 at p . Let us now consider the distribution of the orthogonal complement E_1^\perp , which is well-defined, smooth and integrable in a neighborhood of p . The restriction of the metric g onto E_1^\perp is non-degenerate and is positively definite in one point. Hence, by continuity it is positively definite in a whole neighborhood. Similarly, at a minimum point q one can consider the restriction of (14) to E_1 :

$$\mu_{,ij}|_{E_1} = 2 a_{ij}|_{E_1} = 2 g_{ij}|_{E_1},$$

since $a_{ij}|_{E_1} = \delta_{ij}|_{E_1}$, which implies that g is positively definite on E_1 at q . Considering the distribution E_0^\perp , we obtain that the restriction of g to E_0^\perp is positively definite in a neighborhood of q . Let us now consider the general point x , where

$$T_x M = E_1 \oplus E_0 \oplus \text{span}\{f^i, \bar{f}^i\}.$$

We choose a piecewise smooth path $\gamma : [0, 1] \rightarrow M$, connecting a point $p = \gamma(0)$ where $\mu(p) = 1$ with the point $x = \gamma(1)$, such that there is no point along γ where $\mu = 0$. Since the distribution E_1^\perp is differentiable on $M \setminus \{q \in M : \mu(q) = 0\}$, there cannot be a change of signature of the restriction of the metric $g|_{E_1^\perp}$ along that path, unless the determinant of $g|_{E_1^\perp}$ vanishes at some point between p and x . Since g is nondegenerate, this never can happen and we obtain that $g|_{E_1^\perp}$ is positively definite at x . Exactly the same arguments can be used if one wants to show that $g|_{E_0^\perp}$ is positively definite at x . In the end, we obtain that g is positively definite at each point of M and hence, Theorem 3 is proven.

5. PROOF OF THEOREM 1

Let (M, g, J) be a closed, connected pseudo-Riemannian Kähler manifold and f a non-constant solution of equation (1). By Theorem 2, $c \neq 0$ and the metric g can be replaced by $\tilde{g} = c \cdot g$ without changing the Levi-Civita connection. The function f is now a non-constant solution of (1) with $c = 1$ and using Theorem 3 we obtain that \tilde{g} has to be positive-definite. Applying the classical result of Tanno [12] for positive-definite metrics, we obtain that $(M, 4cg, J)$ has constant holomorphic sectional curvature equal to 1, hence Theorem 1 is proven.

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